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On the existence of indiscernible trees

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ABSTRACT

We introduce several concepts concerning the indiscernibility of trees. A tree is by definition an ordered set $(O, <)$ such that, for any $a \in O$, the initial segment $\{b \in O\colon b < a\}$ determined by a is a linearly ordered set. A typical example of a tree is the set $\omega^{<\omega}$ of finite ω-sequences with the order relation $<_{\text{ini}}$, where $\eta <_{\text{ini}} \nu$ means that η is a proper initial segment of ν. In this paper, we consider some structure M in the language L and are interested in sets A of the form $(a_\eta)_{\eta\in O}$, where O is a tree, and a_η labeled by η is an element in M. Such a set A is also called a tree in this paper. We study the indiscernibility of trees A in general settings and apply the obtained results to the study of unstable theories.

1. Introduction

In model theory, the study of indiscernible sequences is very important. These sequences are used for constructing models, and are also used for analyzing a given model. Fortunately, there is an almost unique definition of the indiscernibility of a sequence. However, different definitions of the indiscernibility of a set labeled by a tree are used for different purpose.

Roughly speaking, $A = (a_\eta)_{\eta\in O}$ is called an indiscernible tree if whenever X and Y are subsets of O having a similar shape (as ordered sets), then the two sets $(a_\eta)_{\eta\in X}$ and $(a_\eta)_{\eta\in Y}$ have the same L-type. Depending on the definition of similarity, we have a number of different definitions of indiscernibility. Among such, Shelah's tree indiscernibility is of particular importance. He thinks of a tree $O = \lambda^{<\omega}$ (and its subtree) as a structure with the predicates $P_n = \{\eta \in O\colon len(\eta) = n\}$ $(n \in \omega)$, the lexicographic order, the order of being an initial segment and the binary meet operator (giving the longest common initial segment). He defines his similarity $(X \sim Y)$ by $atp(X) = atp(Y)$ (X and Y have the same atomic type in this language). In this setting, the following is one of the most important existence results:

Fact. *Let $m, n \in \omega$. Let $O = \lambda^n$ and f be a function from O^m to κ. If the cardinal λ is large enough (compared with κ), then we have an infinitely branching subtree O_0 of the same height such that any two similar sets (ordered properly) of cardinality m have the same f-value. (See Fact 9 and [8, p. 662].)*

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One can use this fact to prove the existence of indiscernible trees satisfying some prescribed condition expressed by L-formulas. In the present paper, the indiscernibility in the sense of Shelah will be referred to as weak indiscernibility. By weakening Shelah's similarity relation, alternate versions of indiscernibility (including strong indiscernibility) will be introduced. There are several papers ([1,4–6] and [7]) concerning tree indiscernibility; however, their approaches are different from that in the present paper.

Let $\Gamma = \Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ denote a set of L-formulas with free variables from $(x_\eta)_{\eta\in\omega^{<\omega}}$. We impose some homogeneity conditions on Γ. Among these conditions are the weak subtree property, the subtree property and the strong subtree property. It is known that if Γ has the weak subtree property, then there exists a weakly indiscernible tree realizing Γ. This has been proven in [8], although not stated explicitly. By assuming a stronger homogeneity condition, we prove the existence of $A \models \Gamma$ satisfying a stronger indiscernibility condition. Among other results, we prove that if Γ has the strong subtree property then Γ is realized by a strongly indiscernible tree.

If the theory T has the tree property (the negation of simplicity, see [9]), there exists a formula $\varphi(x, y)$, $k \in \omega$ and a set $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that (1) $\{\varphi(x, a_{\eta|n}) : n \in \omega\}$ is consistent for each path $\eta \in \omega^\omega$ and (2) for each $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta^\frown\langle n\rangle}) : n \in \omega\}$ is k-inconsistent. The condition for $(a_\eta)_{\eta\in\omega^{<\omega}}$ to satisfy (1) and (2) can be expressed by a set $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ of L-formulas. This particular Γ has the weak subtree property, so it is realized by a weakly indiscernible tree. However, in some cases, we want stronger indiscernibility when studying the tree property.

In Section 4, we discuss indiscernible trees where the labeling tree $O \subset \omega^{<\omega}$ is not infinitely branching. More precisely, we treat the case where every $\eta \in O$ of even length has exactly one child. Such trees are necessary for the study of simple theories (and related theories), which are characterized by the non-existence of a certain type of trees.

The final section, Section 5, discusses applications. We apply the obtained results to the study of unstable theories. First, for showing the usefulness of our results, we give a proof of Shelah's result [8, p. 146] concerning the tree property and the number of independent partitions. We also investigate the relationship between weak-TP_{k+1} and weak-TP_k, which are concepts introduced in [6]. Finally, we show a stronger version of the fact that there is no simple nonlow theory T such that $D_{\mathrm{inp}} < \omega$ (see Definition 35).

2. Weakly indiscernible trees

First we explain some notations we use. Let S be a linearly ordered set. Recall that an initial segment of S is a subset $S_0 \subset S$ such that if $s < t \in S_0$ then $s \in S_0$. The set of all functions $\eta : S_0 \to \alpha$, with S_0 a proper initial segment of S, will be denoted by $\alpha^{<S}$. $\alpha^{<S}$ becomes a tree by $<_{\mathrm{ini}}$, the order relation of being an initial segment: $\eta <_{\mathrm{ini}} \nu$ iff $\eta \neq \nu$ and $\nu|\mathrm{dom}(\eta) = \eta$. A function $\eta : S \to \alpha$ is called a path of $\alpha^{<S}$. We are mainly interested in trees O of the form $\alpha^{<\beta}$, where α and β are ordinals. The elements in O are usually denoted by η or ν.

We work in the monster model $\mathcal{M}$ of the fixed complete theory T formulated in the language L. O is not an object in $\mathcal{M}$. The finite tuples of $\mathcal{M}$ are denoted by $a, b, \ldots$. Small subsets of $\mathcal{M}$ are denoted by $A, B, \ldots$. We are interested in subsets of $\mathcal{M}$ whose elements are labeled by elements in some tree O. For denoting finite sets of O, we use $X, Y, \ldots$. We assume such a set X is enumerated in $<_{\mathrm{lex}}$-increasing order, unless stated otherwise. L-formulas are denoted by $\varphi, \psi, \ldots$. We simply write $\varphi \in L$ if φ is an L-formula. Γ always denote a set of L-formulas (possibly with parameters from $\mathcal{M}$). $\mathrm{tp}(a/A)$ is the complete type of a over A. $S(A)$ is the set of all complete types over A.

For simplicity, definitions below are given for $O = \omega^{<\omega}$.

Definition 1.

1. Let $L_s = \{<_{\mathrm{ini}}, <_{\mathrm{lex}}, \cap, <_{\mathrm{len}}, (P_n)_{n\in\omega}\}$. We consider the following structure on $\omega^{<\omega}$: For $\eta, \nu \in \omega^{<\omega}$,
   (a) $\eta <_{\mathrm{ini}} \nu \Leftrightarrow \eta$ is a proper initial segment of ν;
   (b) $\eta <_{\mathrm{lex}} \nu \Leftrightarrow \eta$ is less than ν in the lexicographic order;
   (c) $\eta \cap \nu =$ the longest common initial segment of η and ν;
   (d) $\eta <_{\mathrm{len}} \nu \Leftrightarrow \mathrm{len}(\eta) < \mathrm{len}(\nu)$, where $\mathrm{len}(\eta)$ is the length of the sequence η;
   (e) $P_n(\eta) \Leftrightarrow$ the length of η is n.
2. Let $X, Y \subset \omega^{<\omega}$ be two finite subsets. We say X is equivalent to Y in Shelah's sense, written as $X \sim_s Y$, if X and Y have the same atomic type with respect to L_s.

Definition 2. We say that $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ is a weakly indiscernible tree over B if whenever $X \sim_s Y$ then $\mathrm{tp}(a_X/B) = \mathrm{tp}(a_Y/B)$, where $a_X = (a_\eta)_{\eta\in X}$.

Definition 3. Let σ be an injective map from $\mathrm{dom}(\sigma) \subset \omega^{<\omega}$ to $\omega^{<\omega}$.

1. We say that σ is an L_s-embedding if for every finite tuple $X \subset \mathrm{dom}(\sigma)$ we have $X \sim_s \sigma(X)$.
2. For $A = (a_\eta)_{\eta\in\omega^{<\omega}}$, A_σ is the set $(b_\eta)_{\eta\in\mathrm{dom}(\sigma)}$, where $b_\eta = a_{\sigma(\eta)}$.

In what follows, $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ is a set of L-formulas with free variables among x_η's (and possibly with parameters). If $X \subset \omega^{<\omega}$, $\Gamma|x_X$ denotes the set of formulas in Γ with free variables in x_X.

Remark 4. Let us consider $\omega^{<\omega}$ as an L_s-structure. Let X be a finite subset of $\omega^{<\omega}$ with $|X| = n$. The condition $X \sim_s Y$ is not sufficient for us to have an L_s-embedding sending X to Y, although their heights are definable using P_n's. However, there is an L_s-formula $\theta_X(y_0, \dots, y_{n-1})$ such that the following conditions are equivalent:

1. There is an L_s-embedding $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ with $\sigma(X) = Y$.
2. $\theta_X(Y)$ holds in $\omega^{<\omega}$.

Proof. For Y to satisfy the condition 1, it is necessary that $X \sim_s Y$. The condition $X \sim_s Y$ can be expressed by an L_s-formula (having the free variables $y_0, \dots, y_{n-1}$). Now let us consider the case $X = \{\langle\rangle, \langle i_0\rangle, \langle i_1\rangle\}$ and $Y = \{\langle\rangle, \langle j_0\rangle, \langle j_1\rangle\}$. We assume $X \sim_s Y$. So, by symmetry, we can assume $i_0 < i_1$ and $j_0 < j_1$ as integers. For such Y to satisfy 1, the following conditions are necessary and sufficient:

(a) $i_0 \leqslant j_0 \in \omega$,
(b) $i_1 - i_0 \leqslant j_1 - j_0 \in \omega$.

The condition (a) can be expressed by the formula $\langle i_0\rangle \leqslant_{\text{lex}} \langle j_0\rangle$. By putting $k = i_1 - i_0$, the condition (b) can be expressed by the formula

$$\exists x_0, \dots, x_k \quad \left[\text{“}x_i\text{'s are immediate successors of } \langle\rangle\text{”} \wedge \langle j_0\rangle = x_0 <_{\text{lex}} x_1 <_{\text{lex}} \cdots <_{\text{lex}} x_k = \langle j_1\rangle\right].$$

So, for this special case, we have shown the existence of a formula θ_X giving the equivalence of 1 and 2. The general case can be proven by the induction on $n = |X|$. □

In subsequent sections, we introduce other tree languages including L_0 and L_1. L_0 and L_1 may be substituted for L_s in the above claim, and we retain an equivalence of 1 and 2, by choosing an appropriate $\theta_X(Y)$.

Definition 5. We say that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the weak subtree property if there is a realization $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ such that if $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ is an L_s-embedding then $A_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes Γ.

Lemma 6. *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ have the weak subtree property. Let λ be an infinite cardinal. Then there is $B = (c_\eta)_{\eta\in\lambda^{<\omega}}$ such that if $\sigma : \omega^{<\omega} \to \lambda^{<\omega}$ is an L_s-embedding then B_σ realizes Γ.*

Proof. We can assume λ is uncountable. Let $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ be a realization of Γ witnessing the weak subtree property of Γ. Let M be a model containing A. We prepare a new unary predicate symbol U with the interpretation $U^M = A$. We regard M as an $(L \cup L_s \cup \{U\})$-structure. Now let N be a sufficiently saturated $(L \cup L_s \cup \{U\})$-elementary extension of M. We choose a subset $B = (b_\eta)_{\eta\in\lambda^{<\omega}}$ of U^N such that, for any $\eta, \nu \in \lambda^{<\omega}$,

1. $\eta \in \omega^{<\omega} \Rightarrow b_\eta = a_\eta$.
2. $N \models P_n(b_\eta) \iff len(\eta) = n (n \in \omega)$.
3. $N \models b_\eta <_{\text{ini}} b_\nu \iff \eta <_{\text{ini}} \nu$.
4. $N \models b_\eta <_{\text{lex}} b_\nu \iff \eta <_{\text{lex}} \nu$.
5. $N \models b_\eta \cap b_\nu = b_{\eta\cap\nu}$.

The conditions 2–5 simply say that the mapping $\eta \mapsto b_\eta$ is an L_S-embedding. Using the weak subtree property, it can be easily seen that M has the following property: For any $\cap$-closed finite $X \subset \omega^{<\omega}$ and $\varphi(x_X) \in \Gamma$,

(*) if there is an L_s-embedding $\tau : \omega^{<\omega} \to U$ sending X to Y, then $\varphi(b_Y)$ holds.

Since N is an elementary extension and since the property (*) can be expressed by an $(L \cup L_s \cup \{U\})$-sentence (using θ_X in Remark 4), the above property is true even if Y is a subset of $\lambda^{<\omega}$. Let $\sigma : \omega^{<\omega} \to \lambda^{<\omega}$ be an arbitrary L_s-embedding. Then $b_{\sigma(X)}$ satisfies θ_X. So $b_{\sigma(X)}$ satisfies $\varphi(x_X) \in \Gamma$. Hence $B_\sigma = (b_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes Γ. □

Example 7. Let $k \in \omega \setminus \{0, 1\}$. T is said to have the k-tree property, in short *k-TP* (see [6]), if there is a formula $\varphi(y, x)$ and a set $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that (1) $\{\varphi(y, a_{\eta|n}) \colon n \in \omega\}$ is consistent for each path $\eta \in \omega^\omega$ and (2) for each $\eta \in \omega^{<\omega}$ the set $\{\varphi(y, a_{\eta^\frown\langle n\rangle}) \colon n \in \omega\}$ is k-inconsistent. The condition for $(a_\eta)_{\eta\in\omega^{<\omega}}$ to satisfy (1) and (2) can be expressed by a set $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ of L-formulas. This Γ has the weak subtree property.

Our goal of this section is the following theorem, which is implicit in [8].

Theorem 8. *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of $L(B)$-formulas. If Γ has the weak subtree property, then Γ is realized by a weakly indiscernible tree over B.*

The following fact (Theorem 2.6 of [8, p. 662]) is essential, for proving the theorem above.

Fact 9 *(Shelah). Let $O = \lambda^{<n}$ be a tree, and $f : O^k \to \mu$ a k-palace function. If λ is sufficiently large (depending only on μ), then there is an L_s-embedding $\sigma : \omega^{<n} \to \lambda^{<n}$ such that $f(\sigma(X)) = f(\sigma(Y))$ for any k-tuples $X, Y \subset \omega^{<n}$ with $X \sim_s Y$.*

In the original statement in Theorem 2.6 of [8, p. 662], λ depends on n, k as well as μ. So λ can be written as $\lambda_{n,k}$. However, by taking $\sup_{n,k\in\omega} \lambda_{n,k}$, we may assume that λ depends only on μ.

Proof of Theorem 8. It is enough to show the following claim:

Claim A. *For any $n \in \omega$, $\Gamma \cup \Delta_n$ is consistent, where $\Delta_n = \{\psi(x_X) \leftrightarrow \psi(x_Y)\colon X, Y \subset \omega^{<n}, X \sim_s Y$ and $\psi \in L(B)\}$.*

Take any finite subset Δ of Δ_n. Let k be a number such that if $\psi(x_X) \leftrightarrow \psi(x_Y)$ belongs to Δ then $|X| = |Y| \leqslant k$. For $\mu = 2^{|L(B)|}$, we choose a sufficiently large λ satisfying the condition mentioned in Fact 9. Then, by Lemma 6, we can choose $A = (a_\eta)_{\eta\in\lambda^{<\omega}}$ such that if $\sigma : \omega^{<\omega} \to \lambda^{<\omega}$ is an L_s-embedding then A_σ realizes Γ. Let $f : (\lambda^{<n})^k \to S^k(B)$ be the function defined by

$$(\eta_1, \ldots, \eta_k) \mapsto \mathrm{tp}(a_{\eta_1}, \ldots, a_{\eta_k}/B).$$

For this f, we apply Fact 9 and get an embedding $\sigma : \omega^{<n} \to \lambda^{<n}$ such that $f(\sigma(X)) = f(\sigma(Y))$ for any k-tuples $X, Y \subset \omega^{<n}$ with $X \sim_s Y$. Then the set A_σ realizes Δ as well as Γ. So we have shown the finite satisfiability of $\Gamma \cup \Delta_n$ and we are done. □

Remark 10.

1. Let $\Gamma^* = \{\varphi(x_{\sigma(X)})\colon \varphi(x_X) \in \Gamma,\ \sigma$ an L_s-embedding$\}$. Then A realizes Γ^* if and only if A witnesses the weak subtree property of Γ.
2. In [7], they define the set $EM_s(A) = \{\varphi(x_X)\colon \mathcal{M} \models \varphi(a_Y)$ for all $Y \sim_s X\}$ of L-formulas and prove that for all $A = (a_\eta)_\eta$ there is a weak indiscernible tree (in our sense) realizing $EM_s(A)$ (see Remark 3.14 in [7]). $EM_s(A)$ has the weak subtree property.

3. Indiscernible trees and strongly indiscernible trees

Let $L_0 = \{<_{\mathrm{lex}}, <_{\mathrm{ini}}, \cap\}$ and $L_1 = L_0 \cup \{<_{\mathrm{len}}\}$. The (0)-similarity $\sim_0$ and the (1)-similarity $\sim_1$ are defined in a similar way to $\sim_s$.

Definition 11. Let $i \in \{0, 1\}$. Let $X, Y \subset \omega^{<\omega}$ be two finite subsets. We say X is (i)-similar to Y, in symbol $X \sim_i Y$, if X and Y have the same L_i-atomic type.

Definition 12. Let $i \in \{0, 1\}$. We say that $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ is an (i)-indiscernible tree over B if whenever $X \sim_i Y$ then $\mathrm{tp}(a_X/B) = \mathrm{tp}(a_Y/B)$. The (1)-indiscernibility is referred as the indiscernibility, and the (0)-indiscernibility is referred as the strong indiscernibility.

Remark 13.

1. $L_0 \subset L_1 \subset L_s$.
2. If A is a strongly indiscernible tree then A is an indiscernible tree.
3. If A is an indiscernible tree, then A is a weakly indiscernible tree.

The notion of L_i-embeddings is defined naturally.

Definition 14. Let $i \in \{0, 1\}$. We say that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the (i)-subtree property, if there is a set $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ such that if $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ is an L_i-embedding then the set $A_\sigma = (a_{\sigma(\eta)})_{\eta\in X}$ realizes Γ. The (1)-subtree property is referred as the subtree property, and the (0)-subtree property is referred as the strong subtree property.

Notice that the condition $X \sim_s Y$ for finite X and Y is equivalent to

$$X \sim_1 Y \quad \text{and} \quad lev(X) = lev(Y),$$

where $lev(X) = \{len(\eta)\colon \eta \in \mathrm{cl}(X)\}$, and $\mathrm{cl}(X)$ is the $\cap$-closure of X. This equivalence will be used in our proof of the following theorem.

Theorem 15. *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of $L(B)$-formulas. If $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the subtree property, then Γ is realized by an indiscernible tree over B.*

Proof. For simplicity, we assume $B=\emptyset$. So to prove this theorem it is sufficient to prove the following.

Claim A. *Let X be a finite $\cap$-closed set and let $\varphi_1(x_X),\dots,\varphi_n(x_X)$ be a finite number of L-formulas. Let $\Delta=\{\varphi_i(x_{Y_1})\leftrightarrow\varphi_i(x_{Y_2}):i=1,\dots,n,\ Y_1\sim_1 Y_2\sim_1 X\}$. Then $\Gamma\cup\Delta$ is consistent.*

Since the subtree property implies the weak subtree property, by Theorem 8, we have a weakly indiscernible tree $A=(a_\eta)_{\eta\in\omega^{<\omega}}$ realizing Γ. Let $k=|lev(X)|$. For each formula $\varphi=\varphi_i(x_X)$, we can define a mapping $f_\varphi:[\omega]^k\to\{0,1\}$ by

$$f_\varphi(\{n_0,\dots,n_{k-1}\})=1$$

if and only if $\varphi(a_Y)$ holds for some (any) $Y\sim_1 X$ with $lev(Y)=\{n_0,\dots,n_{k-1}\}$. By Ramsey's theorem, there is an infinite set $H\subset\omega$ such that f_φ is constant on $[H]^k$. Let $\{h_i\colon i\in\omega\}$ be the enumeration of H in increasing order. For a sequence $\eta=\langle\eta(0),\dots,\eta(l-1)\rangle\in\omega^{<\omega}$ of length l, we define $\sigma_H(\eta)\in\omega^{<\omega}$ of length h_l by

$$\sigma_H(\eta)=0^{h_0}{}^\frown\eta(0)^{h_1-h_0}{}^\frown\eta(1)^{h_2-h_1}{}^\frown\cdots{}^\frown\eta(l-1)^{h_l-h_{l-1}},$$

where x^l denotes the l-time iteration of x. Then $\sigma_H:\omega^{<\omega}\to\omega^{<\omega}$ is an L_1-embedding with $lev(\mathrm{ran}(\sigma_H))=H$. So A_{σ_H} realizes Γ. Moreover, by our choice of H, the set $(b_\eta)_{\eta\in\omega^{<\omega}}:=A_{\sigma_H}=(a_{\sigma_H(\eta)})_{\eta\in\omega^{<\omega}}$ is a $\varphi(x_X)$-indiscernible tree in the following sense:

$$(*)\ \ Y_1,Y_2\subset\omega^{<\omega},\ X\sim_1 Y_i\ \ (i=1,2)\ \Rightarrow\ \models\varphi(b_{Y_1})\leftrightarrow\varphi(b_{Y_2}).$$

The above argument shows the consistency of $\Gamma\cup\Delta$. $\square$

Theorem 16. *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of $L(B)$-formulas. If $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the strong subtree property, then Γ is realized by a strongly indiscernible tree over B.*

Proof. We assume $B=\emptyset$. By Theorem 15, we have an indiscernible tree realizing Γ. So, by compactness, there is an indiscernible tree $A=(a_\eta)_{\eta\in\omega^{<\omega_1}}$ such that if $\sigma:\omega^{<\omega}\to\omega^{<\omega_1}$ is an L_1-embedding then A_σ realizes Γ.

Claim A. *For each $n\in\omega$, there is an L_0-embedding $\sigma_n:\omega^{<n}\to\omega^{<\omega_1}$ such that if $\eta<_{\mathrm{lex}}\nu\in\mathrm{dom}(\sigma_n)$ then $\mathrm{len}(\sigma_n(\eta))<\mathrm{len}(\sigma_n(\nu))$.*

We prove the claim by induction on n. Let $\sigma_0(\langle\rangle)=\langle\rangle$ and suppose that we have defined σ_n from $\omega^{<n}$ to $\omega^{<\omega_1}$ such that if $\eta<_{\mathrm{lex}}\nu$ then $\sigma_n(\eta)<_{\mathrm{len}}\sigma_n(\nu)$. Since the cofinality of ω_1 is $>\omega$, there is $\alpha_0<\omega_1$ such that the lengths of $\sigma_n(\eta)$ $(\eta\in\mathrm{dom}(\sigma_n))$ are all less than α_0. Now we define σ_{n+1} by the equation

$$\sigma_{n+1}\big(\langle i\rangle^\frown\eta\big)=\underbrace{\langle i,i,\dots\rangle}_{\alpha_0\cdot(i+1)}{}^\frown\sigma_n(\eta).$$

This definition implies that $\alpha_0\cdot(i+1)\leqslant len(\sigma_{n+1}(\langle i\rangle^\frown\eta))<\alpha_0\cdot(i+2)$. So, in particular, we have $len(\sigma_{n+1}(\langle i\rangle^\frown\eta))<len(\sigma_{n+1}(\langle i'\rangle^\frown\eta'))$, if $i<i'$. By induction on the length of η, we can prove

$$\sigma_{n+1}(\eta^\frown\nu)=\sigma_{n+1}(\eta)^\frown\sigma_{n+1-len(\eta)}(\nu), \tag{*}$$

if $\eta^\frown\nu\in\mathrm{dom}(\sigma_{n+1})$. So, σ_{n+1} is an L_0-embedding. Now we show that:

$$\eta<_{\mathrm{lex}}\eta'\quad\Rightarrow\quad\sigma_{n+1}(\eta)<_{\mathrm{len}}\sigma_{n+1}(\eta'). \tag{**}$$

For proving the condition (**), let $\nu=\eta\cap\eta'$. If $\eta<_{\mathrm{ini}}\eta'$ (i.e. $\nu=\eta$), then clearly we have $\sigma_{n+1}(\eta)<_{\mathrm{len}}\sigma_{n+1}(\eta')$. So we can assume $len(\nu)>0$, $\eta=\nu^\frown\langle i\rangle^\frown\eta_0$, $\eta'=\nu^\frown\langle i'\rangle^\frown\eta_0'$, and $i<i'$. By $(*)$, using the induction hypothesis, we have

$$\begin{aligned}len\big(\sigma_{n+1}(\eta)\big)&=len\big(\sigma_{n+1}(\nu)\big)+len\big(\sigma_{n+1-len(\nu)}\big(\langle i\rangle^\frown\eta_0\big)\big)\\&<len\big(\sigma_{n+1}(\nu)\big)+len\big(\sigma_{n+1-len(\nu)}\big(\langle i'\rangle^\frown\eta_0'\big)\big)\\&=len\big(\sigma_{n+1}(\eta')\big).\end{aligned}$$

Thus the condition (**) was shown, and σ_{n+1} has the required property. We have shown the existence of σ_n's for all n. (End of proof of Claim A.)

To complete our proof of the theorem, it is enough to show the following claim:

Claim B. *$\Gamma\cup\Delta$ is consistent, where $\Delta=\{\varphi(x_X)\leftrightarrow\varphi(x_Y)\colon X,Y\subset\omega^{<\omega},X\sim_0 Y$ and $\varphi\in L\}$.*

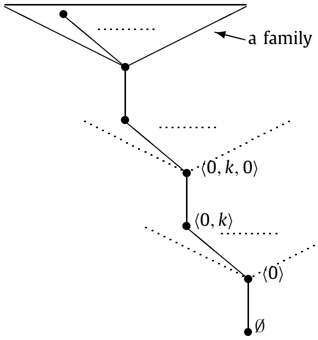

Fig. 1. A figure of O.

Fix $n \in \omega$, and let $\sigma : \omega^{<n} \to \omega^{<\omega_1}$ be the L_0-embedding given in Claim A. Then $A_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<n}}$ realizes $\Gamma|(x_\eta)_{\eta\in\omega^{<n}}$. Moreover, if $X \sim_0 Y \subset \omega^{<n}$ then $\sigma(X) \sim_1 \sigma(Y)$. So, A_σ realizes $(\Gamma \cup \Delta)|(x_\eta)_{\eta\in\omega^{<n}}$ because A is an indiscernible tree. Finally, using a compactness argument, we can show that $\Gamma \cup \Delta$ is finitely satisfiable. $\square$

Consider the language $\{<_{\text{lex}}, <_{\text{ini}}\}$, which is weaker than L_0. The following example shows that we cannot hope to have a $\{<_{\text{lex}}, <_{\text{ini}}\}$-version of Theorem 16.

Example 17. Let $L = \{<_{\text{lex}}, <_{\text{ini}}\}$. We consider $M = \omega^{<\omega}$ as an L-structure. Then, in $T = \mathrm{Th}_L(M)$,

$$\Gamma = \left\{x_\eta <_{\text{ini}} x_\nu \colon \eta <_{\text{ini}} \nu \in \omega^{<\omega}\right\} \cup \left\{x_\eta \not<_{\text{ini}} x_\nu \colon \eta \not<_{\text{ini}} \nu \in \omega^{<\omega}\right\} \cup \left\{x_\eta <_{\text{lex}} x_\nu \colon \eta <_{\text{lex}} \nu \in \omega^{<\omega}\right\}$$

has the subtree property with respect to L. Namely, if $\sigma : M \to M$ is an L-preserving mapping, then $\sigma(M)$ satisfies Γ. We claim that no realization of Γ is an L-indiscernible. Let $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ be a realization of Γ. Let us consider $X = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$ and $Y = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$. Clearly $X \sim_L Y$. However, since the meet operator $\cap$ is definable in T, a_X and a_Y do not have the same L-type. For instance, we have $a_{\langle 0,0\rangle} \cap a_{\langle 1,1\rangle} = a_{\langle 0,1\rangle} \cap a_{\langle 1,1\rangle}$ and $a_{\langle 0,0\rangle} \cap a_{\langle 1,1\rangle} \neq a_{\langle 1,0\rangle} \cap a_{\langle 1,1\rangle}$. Hence A is not $\{<_{\text{lex}}, <_{\text{ini}}\}$-indiscernible.

4. Indiscernible trees in other settings

In this section, we study different versions of indiscernibility. Throughout this section, we are mainly interested in $O = \{\eta \in \omega^{<\omega} : \eta(2n) = 0 \text{ for all } n \in \omega\}$. If $\eta \in O$ then it has the form

$$\eta = \langle 0, \eta(1), 0, \eta(3), 0, \ldots, \eta(n-1)\rangle,$$

where $len(\eta) = n$ (see Fig. 1). Of course, if η is of odd length ($n-1$ is even), then $\eta(n-1) = 0$. O is closed under taking the operator $\cap$ (in $\omega^{<\omega}$). So, we can impose an L_s-structure on O as a substructure of $\omega^{<\omega}$.

We call a set $\{\eta\} \cup \{\eta^\frown\langle n\rangle \colon n \in \omega\} \subset O$ a *family* if $\eta \in O$ has odd length. We need to consider the *family relation* $F(\eta_1, \eta_2)$, the relation $E(\eta)$ designating the even length elements, and the *family order* $\eta_1 <_F \eta_2$ on O defined by the following:

- $F(\eta_1, \eta_2) \iff \eta_1$ and η_2 belong to the same family;
- $E(\eta) \iff len(\eta)$ is even;
- $\eta_1 <_F \eta_2 \iff len(\eta_1) \leqslant 2n < len(\eta_2)$ for some $n \in \omega$.

$\eta_1 <_F \eta_2$ means that the family of η_1 is "older" than that of η_2. We will write $\eta_1 =_F \eta_2$ if η_1 and η_2 are the same "generation", i.e.,

$$\eta_1 \not<_F \eta_2 \quad \text{and} \quad \eta_2 \not<_F \eta_1,$$

equivalently $\{len(\eta_1), len(\eta_2)\} \subset \{2n, 2n-1\}$ for some $n \in \omega \setminus \{0\}$.

Definition 18. The tree languages for O we will consider in this section are:

- $L_{0,F} = L_0 \cup \{F, E\} = \{<_{\text{ini}}, <_{\text{lex}}, \cap\} \cup \{F, E\}$;
- $L_{1,F} = L_1 \cup \{F, E, <_F\} = \{<_{\text{ini}}, <_{\text{lex}}, \cap, <_{\text{len}}\} \cup \{F, E, <_F\}$;
- $L_{\text{s},F} = L_{\text{s}} \cup \{F, E\} = \{<_{\text{ini}}, <_{\text{lex}}, \cap, <_{\text{len}}, (P_n)_{n\in\omega}\} \cup \{F, E\}$.

For $* \in \{\text{s}, 0, 1\}$, the $L_{*,F}$-similarity ($\sim_{*,F}$) and the $L_{*,F}$-indiscernibility of $(a_\eta)_{\eta\in O}$ are defined similarly as before.

Example 19.

1. $X \sim_{1,F} \{\eta^\frown\nu : \nu \in X\}$, for any $X \subset O$ and any $\eta \in O$ of even length.
2. $\langle 0\rangle \nsim_{0,F} \langle 0, i\rangle$.
3. $\langle 0\rangle, \langle 0, i\rangle \nsim_{0,F} \langle 0\rangle, \langle 0, i, 0, j\rangle$.
4. $\langle 0, i\rangle, \langle 0, j, 0, k\rangle \nsim_{0,F} \langle 0, i, 0, l\rangle, \langle 0, j, 0, k\rangle$.

Definition 20. We say $\Gamma((x_\eta)_{\eta\in O})$ has $L_{s,F}$-subtree property if there is a realization $A \models \Gamma$ such that for every $L_{s,F}$-embedding $\sigma : O \to O$ the image A_σ realizes Γ.

Theorem 21. *Suppose $\Gamma((x_\eta)_{\eta\in O})$ has the $L_{s,F}$-subtree property. Then Γ is realized by an $L_{s,F}$-indiscernible tree.*

Proof. Let $A = (a_\eta)_{\eta\in O}$ be a realization of Γ witnessing the $L_{s,F}$-subtree property. For each $\eta \in \omega^{<\omega}$ of length l, let $\eta^* \in O$ be the sequence $\langle 0, \eta(0), \ldots, 0, \eta(l-1)\rangle$. We now define a new tree. For $\eta \in \omega^{<\omega}$, let

$$y_\eta := x_{\eta^{*-}}, x_{\eta^*},$$

where η^{*-} is the immediate predecessor of η^* in O. Then we regard Γ as a set of formulas with free variables among y_η's. If we put $b_\eta = a_{\eta^{*-}}, a_{\eta^*}$, then $B = (b_\eta)_{\eta\in\omega^{<\omega}}$ witnesses the L_s-subtree property of $\Gamma((y_\eta)_{\eta\in\omega^{<\omega}})$. So, there is a weakly indiscernible tree $B' = (b'_\eta)_{\eta\in\omega^{<\omega}}$ realizing $\Gamma((y_\eta)_{\eta\in\omega^{<\omega}})$. By letting $a'_{\eta^{*-}}$ be the first coordinate of b'_η and letting a'_{η^*} the second coordinate, we see that $(a'_\eta)_{\eta\in O}$ is an $L_{s,F}$-indiscernible tree realizing $\Gamma((x_\eta)_{\eta\in O})$. $\square$

Definition 22. We say $\Gamma((x_\eta)_{\eta\in O})$ has $L_{i,F}$-subtree property if there is a realization $A \models \Gamma$ such that for every $L_{i,F}$-embedding $\sigma : O \to O$ the image A_σ realizes Γ.

Definition 23. Let $H = \{h_i\colon\ i \in \omega\} \subset \omega$ be an infinite set of even numbers enumerated in the increasing order. We define a map $\tau_H : O \to O$ by

$$\tau_H(\eta) = \begin{cases} \langle \underbrace{0, \ldots, 0}_{h_0+1}, \eta(1), \underbrace{0, \ldots, 0}_{h_1-(h_0+1)}, \eta(3), \ldots, \underbrace{0, \ldots, 0}_{h_{m-1}-(h_{m-2}+1)} \rangle & l \text{ is odd,} \\ \langle \underbrace{0, \ldots, 0}_{h_0+1}, \eta(1), \underbrace{0, \ldots, 0}_{h_1-(h_0+1)}, \eta(3), \ldots, \underbrace{0, \ldots, 0}_{h_{m-1}-(h_{m-2}+1)}, \eta(l-1) \rangle & l \text{ is even,} \end{cases}$$

where l is the length of η and m is the integer part of $l/2$. (We stipulate $\tau_H(\langle\rangle) = 0^{h_0}$ and $\tau_H(\langle 0\rangle) = 0^{h_0+1}$.) We put $O_H = \tau_H(O)$.

For example, if $H = \{0, 4, 6, \ldots\}$, then $\tau_H(\langle 0, 1, 0\rangle) = \langle 0, 1, 0, 0, 0\rangle \in \omega^5$, $\tau_H(\langle 0, 1, 0, 2\rangle) = \langle 0, 1, 0, 0, 0, 2\rangle \in \omega^6$ and $\tau_H(\langle 0, 1, 0, 2, 0, 3\rangle) = \langle 0, 1, 0, 0, 0, 2, 0, 3\rangle \in \omega^8$.

Remark 24.

1. τ_H is an $L_{1,F}$-embedding.
2. If $\eta \in O_H$ then $len(\eta) = h + 1$ or $h + 2$ for some $h \in H$. If H is the set of all even numbers, then τ_H is the identity mapping.

Theorem 25. *Suppose $\Gamma((x_\eta)_{\eta\in O})$ has the $L_{1,F}$-subtree property. Then Γ is realized by an $L_{1,F}$-indiscernible tree.*

Proof. Choose an $L_{s,F}$-indiscernible tree $A = (a_\eta)_{\eta\in O}$ realizing Γ. For finite $X \subset O$, let $\mathrm{cl}(X)$ be the $\cap$-closure of X. In the present proof, the level set $lev(X)$ of X is the set

$$\big\{n \in 2\mathbb{N}\colon\ n = len(\eta) - 1 \text{ or } len(\eta) - 2 \text{ for some } \eta \in \mathrm{cl}(X)\big\}.$$

Clearly $lev(X)$ is a subset of H. We fix a finite X.

Claim A. *For any $Y \sim_{1,F} X$ with the same level set as X, and for any formula $\varphi(x_X)$, we have*

$$\models \varphi(a_X) \leftrightarrow \varphi(a_Y).$$

It is enough to show that $X \sim_{s,F} Y$, because of $L_{s,F}$-tree indiscernibility. Note that $lev(X) \subset lev(Y)$ if and only if for all $\eta \in \mathrm{cl}(X)$ there is $\nu \in \mathrm{cl}(Y)$ such that $\eta =_F \nu$. Let $\mathrm{cl}(X) = \{\eta_0, \dots, \eta_{k-1}\} \sim_{1,F} \{\nu_0, \dots, \nu_{k-1}\} = \mathrm{cl}(Y)$ and $\nu_i \leqslant_{\mathrm{len}} \nu_{i+1}$. Suppose $X \sim_{s,F} Y$ is not the case. Then $len(\eta_i) \neq len(\nu_i)$ for some $i < k$. Let i_0 be the minimum such i and assume $len(\eta_{i_0}) < len(\nu_{i_0})$, by symmetry. By the $(1, F)$-similarity, more specifically by the definition of E, $len(\eta_{i_0})$ and $len(\nu_{i_0})$ have the same parity. So, we have $\eta_{i_0} <_F \nu_{i_0}$. Since $lev(X) = lev(Y)$, there is a ν_j such that $\eta_{i_0} =_F \nu_j$. Then, j must be less than i_0 because $len(\nu_j) < len(\nu_{i_0})$. By the minimality of i_0, we have $len(\eta_j) = len(\nu_j)$. Therefore we get $\eta_j =_F \eta_{i_0}$. This is contradictory to $\nu_j <_F \nu_{i_0}$ and $X \sim_{1,F} Y$. (End of Proof of Claim A.)

So, for each $\varphi(x_X)$ with $|lev(X)| = k$, we can define a mapping $f_\varphi : [\omega]^k \to \{0, 1\}$ by:

$$f_\varphi\big(\{n_0, \dots, n_{k-1}\}\big) = 1$$

if and only if $\varphi(a_Y)$ holds for some (any) $Y \sim_{1,F} X$ with $lev(Y) = \{2n_0, \dots, 2n_{k-1}\}$. By Ramsey's theorem, there is an infinite set $G \subset \omega$ such that f_φ is constant on $[G]^k$. In other words, the set $\{a_\eta : \eta \in O_{2G}\}$ is a $\varphi(x_X)$-indiscernible tree in the following sense:

(*) $Y_1, Y_2 \subset O_{2G},\ X \sim_{1,F} Y_i\ (i = 1, 2) \Rightarrow\ \models \varphi(a_{Y_1}) \leftrightarrow \varphi(a_{Y_2})$.

Notice that, for any H of even numbers, $A_{\tau_H} = (a_{\tau_H(\eta)})_{\eta \in O}$ realizes Γ and A_{τ_H} is an $L_{s,F}$-indiscernible tree. By the previous argument, for each finite set $X \subset O$ and each formula $\varphi(x_X)$, we can find $G \subset \omega$ such that $A_{\tau_{2G}}$ becomes a φ-indiscernible tree. Hence, by compactness, we can find $D = (d_\eta)_{\eta \in O}$ realizing Γ such that, if $X \sim_{1,F} Y$ are subsets of O, then d_X and d_Y have the same L-type. □

By a similar argument as above plus the argument of Theorem 16, we can also show the following theorem.

Theorem 26. *Suppose $\Gamma((x_\eta)_{\eta \in O})$ has $L_{0,F}$-subtree property. Γ is realized by an $L_{0,F}$-indiscernible tree.*

Proof. It is sufficient to construct an $L_{0,F}$-embedding $\tau_n : O \cap \omega^{<n} \to \omega^{<\omega_1}$ such that $\eta <_{\mathrm{lex}} \nu \Rightarrow \eta <_{\mathrm{len}} \nu$ if η and ν belong to different families. But such an embedding can be constructed in almost the same way as in Claim A of Theorem 16. □

Example 27. Suppose that T has the k-tree property witnessed by φ. Let $\Gamma((x_\eta)_{\eta \in \omega^{<\omega}})$ be the set in Example 7 expressing this k-tree property. Then Γ does not have the subtree property (in general). So, we cannot expect to have an indiscernible tree realizing Γ. However, the set $\Gamma|(x_\eta)_{\eta \in O}$ has the $L_{0,F}$-subtree property.

5. Some applications

In this section, we will study the tree property and the number of independent partitions.

5.1. Tree property and independent partitions

As a demonstration, we give a proof of Theorem 7.11 in [8, p. 146] using Theorem 26 of the last section.

Fact 28. *T has k-TP if and only if T has 2-TP.*

Proposition 29 *(Shelah). Suppose that T has the tree property and let $\varphi(x, y)$ be a formula witnessing the 2-TP. Then one of the following must hold*:

1. *There is a tree $C = (c_\eta)_{\eta \in \omega^{<\omega}}$ and a formula $\psi = \varphi(x, y_0) \wedge \dots \wedge \varphi(x, y_{m-1})$ with the following properties:*
   (a) *for each path $\eta \in \omega^\omega$, $\{\psi(x, c_{\eta|n}) : n \in \omega\}$ is consistent*;
   (b) *$\psi(x, c_\eta) \wedge \psi(x, c_\nu)$ is inconsistent for any incomparable η and $\nu \in \omega^{<\omega}$.*
2. *There are sets $I_i = (b_{i,j})_{j \in \omega}$ $(i \in \omega)$ with the following properties*:
   (a) *for each path $\eta \in \omega^\omega$, $\{\varphi(x, b_{i,\eta(i)}) : i \in \omega\}$ is consistent*;
   (b) *for each $i \in \omega$, $\{\varphi(x, b_{i,j}) : j \in \omega\}$ is 2-contradictory.*

Proof. Let $\Gamma((x_\eta)_{\eta \in \omega^{<\omega}})$ be the set expressing the 2-*TP* witnessed by φ. Let

$$O_{\mathbb{Z}} = \big\{\eta \in \omega^{<\mathbb{Z}} : \forall n \in \mathbb{Z},\ \eta(2n) = 0\big\},$$
$$O = \big\{\eta \in \omega^{<\omega} : \forall n \in \omega,\ \eta(2n) = 0\big\},$$

and $\Gamma_O = \Gamma|(y_\eta)_{\eta \in O}$. ($\eta \in \omega^{<\mathbb{Z}}$ means that η is a function from $\{k \in \mathbb{Z} : k < m\}$ to ω for some $m \in \mathbb{Z}$.) Clearly Γ_O has the $L_{0,F}$-subtree property. So, by Theorem 26, Γ_O is realized by an $L_{0,F}$-indiscernible tree, say $A = (a_\eta)_{\eta \in O}$. By compactness, we

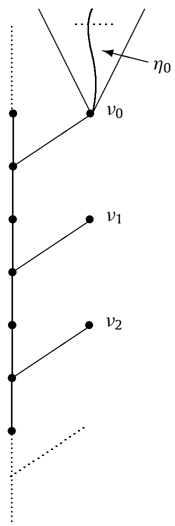

Fig. 2. ν_i and η_i.

may assume that the elements in A are labeled by $\omega^{<\mathbb{Z}}$. So we assume $A = (a_\eta)_{\eta \in O_\mathbb{Z}}$. For $i \in \omega$, let $\nu_i \in O_\mathbb{Z}$ be the function with $\mathrm{dom}(\nu_i) = \{k \in \mathbb{Z}\colon\ k < -2i\}$ defined by $\nu_i(k) = 0$ for all $k < -2i - 1$ and $\nu_i(-2i-1) = 1$(see Fig. 2). Then there are two cases:

- for any set $\{\eta_i\colon\ i \in \omega\}$ of paths of O, $\bigcup_{i\in\omega}\{\varphi(x, a_{\nu_i \frown (\eta_i|n)})\colon\ n \in \omega\}$ is consistent;
- there are paths η_i $(i \in \omega)$ of O such that $\bigcup_{i\in\omega}\{\varphi(x, a_{\nu_i \frown (\eta_i|n)})\colon\ n \in \omega\}$ is inconsistent.

First assume the first case holds. Using ν_i, we define $b_{i,j}$ by

$$b_{i,j} = a_{\nu_i \frown \langle 0, j\rangle}.$$

Notice that, $\{\varphi(x, b_{i,j})\colon\ j \in \omega\}$ is 2-contradictory. So, by the case assumption, we see that the conditions 2(a) and 2(b) are both satisfied.

Then we assume the second case. By compactness, there is a minimal finite set $K \subset \omega$ such that $\{\varphi(x, a_{\nu_i \frown (\eta_i|n)})\colon$ $i \in K,\ n \in \omega\}$ is inconsistent. By the condition of 2-*TP*, we have $|K| \geqslant 2$. Using compactness again, there is an odd number $n_0 \in \omega$ such that $\{\varphi(x, a_{\nu_i \frown (\eta_i|n)})\colon\ i \in K,\ n < n_0\}$ is inconsistent. By the indiscernibility, we assume $K = \{0, 1, \ldots, k-1\}$. Let $\delta(x)$ be the formula $\bigwedge\{\varphi(x, a_{\nu_i \frown \eta_i|n})\colon\ 2 \leqslant i < k,\ n < n_0\}$. Now we work inside the set defined by $\delta(x)$. Let

$$\psi_0(x, y_0, \ldots, y_{n_0-1}) = \varphi(x, y_0) \wedge \cdots \wedge \varphi(x, y_{n_0-1}).$$

To simplify the notation, let $X_i = \{\nu_i \frown (\eta_i|0), \ldots, \nu_i \frown (\eta_i|n_0-1)\}$ $(i < 2)$. Then $\{\psi_0(x, a_{X_0}), \psi_0(x, a_{X_1})\}$ is inconsistent. Now we consider a subtree with the root ν_0. For $i < n_0$ and for $\nu = \langle m_0, \ldots, m_{l-1}\rangle \in \omega^{<\omega}$, put

$$\nu^* = \nu_0 \frown \langle 0, m_0\rangle \frown 0^{n_0+1} \frown \langle 0, m_1\rangle \frown 0^{n_0+1} \frown \cdots \frown \langle 0, m_{l-1}\rangle \frown 0^{n_0+1},$$
$$X_{\nu^*} = \left\{\left(\nu^*\right)^{-k}\colon\ k = 0, \ldots, n_0-1\right\},$$
$$c_\nu = a_{X_{\nu^*}},$$

where 0^l denotes the l-th iteration of 0, and $(\nu^*)^{-k}$ is the k-th predecessor of ν^*. Notice that ν^* is an element of $O_\mathbb{Z}$. Then, for any incomparable ν and $\nu' \in \omega^{<\omega}$, there is no family to which ν^{*-i} and ν'^{*-j} belong $(i, j < n_0)$. This will be used in the proof of Claim B below.

Claim A. *For each path $\eta \in \omega^\omega$, $\{\psi_0(x, c_{\eta|n})\colon\ n \in \omega\}$ is consistent.*

Fix a path $\eta \in \omega^\omega$. There is a path η' of O such that $c_{\eta|n} \subset \{a_{\eta'|m}\colon\ m \in \omega\}$ for every $n \in \omega$. So, the claim follows from the minimality of K and the indiscernibility of A. (End of proof of Claim A.)

Claim B. *$\psi_0(x, c_\eta) \wedge \psi_0(x, c_\nu)$ is inconsistent for any incomparable η and $\nu \in \omega^{<\omega}$.*

Recall $X_i = \{\nu_i{}^\frown(\eta_i|n)\colon n < n_0\}$ $(i = 0, 1)$. Let Y be the set of parameters in δ. Then for any incomparable η and $\nu \in \omega^{<\omega}$ with $\eta <_{\mathrm{lex}} \nu$,

$$X_{\eta^*}, X_{\nu^*}, Y \sim_{L_{0,F}} X_0, X_1, Y,$$

since any element in X_{η^*} and any element in X_{ν^*} are not in the same family. Then, by the $L_{0,F}$-indiscernibility, for any incomparable pair $\eta, \nu \in \omega^{<\omega}$, $\{\psi_0(x, c_\nu), \psi_0(x, c_\eta)\}$ is inconsistent (under $\delta(x)$). (End of proof of Claim B.)

Claim A and Claim B show that $\psi(x, y) = \psi_0(x, y) \wedge \delta(x)$ satisfies the conditions 1(a) and 1(b). □

5.2. Weak TP_1-trees

The following definitions are from [6].

Definition 30. Let $k \in \omega \setminus \{0, 1\}$. T has k-TP_1 if there is a formula $\varphi(x, y)$ and parameters a_η $(\eta \in \omega^{<\omega})$ such that (1) for each path η, $\{\varphi(x, a_{\eta|n})\colon n \in \omega\}$ is consistent and (2) if $\{\nu_0, \ldots, \nu_{k-1}\}$ is a pairwise $<_{\mathrm{ini}}$-incomparable subset of $\omega^{<\omega}$ then $\{\varphi(x, a_{\nu_i})\colon i < k\}$ is inconsistent.

Definition 31. Let $k \in \omega \setminus \{0, 1\}$. T has the weak k-TP_1 if there is a formula $\varphi(x, y)$ and parameters a_η $(\eta \in \omega^{<\omega})$ such that (1) for each path η, $\{\varphi(x, a_{\eta|n})\colon n \in \omega\}$ is consistent and (2) if $\{\nu_0, \ldots, \nu_{k-1}\}$ is a pairwise $<_{\mathrm{ini}}$-incomparable subset of $\omega^{<\omega}$ satisfying $\nu_i \cap \nu_j = \nu_{i'} \cap \nu_{j'}$ for any $i \neq j$ and $i' \neq j'$ then $\{\varphi(x, a_{\nu_i})\colon i < k\}$ is inconsistent.

In [6], they say that $\nu_1, \ldots, \nu_k$ are distant siblings if the condition $\nu_i \cap \nu_j = \nu_{i'} \cap \nu_{j'}$ holds for any $i < j$ and $i' < j'$. If we use this term, the condition (2) in Definition 31 is expressed as follows: if $\{\nu_0, \ldots, \nu_{k-1}\}$ is a family of distant siblings then $\{\varphi(x, a_{\nu_i})\colon i < k\}$ is inconsistent.

Remark 32.

1. Let $\Gamma((y_\eta)_{\eta\in\omega^{<\omega}})$ be the set expressing that $\varphi(x, y)$ witnesses the weak k-TP_1. Then Γ has the strong subtree property.
2. Suppose that $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ and $\varphi(x, y)$ witness the weak k-TP_1. Let $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ be the mapping defined by $\sigma(\langle\rangle) = \langle 0\rangle$ and $\sigma(\eta^\frown\langle i\rangle) = \sigma(\eta)^\frown\langle i, 0\rangle$, and let $b_\eta = a_{\sigma(\eta)}a_{\sigma(\eta)^-}$. Then the new tree $B = (b_\eta)_{\eta\in\omega^{<\omega}}$ and $\varphi(x, y_1) \wedge \varphi(x, y_2)$ also witness the weak k-TP_1.
   For an arbitrary $n \in \omega \setminus \{0\}$, we can define σ_n by $\sigma_n(\eta^\frown\langle i\rangle) = \sigma_n(\eta)^\frown\langle i\rangle^\frown 0^n$. Then, by letting $b_\eta = a_{\sigma_n(\eta)}a_{\sigma_n(\eta)^-}\cdots a_{\sigma_n(\eta)^{-n}}$ and $\psi(x, y_1, \ldots, y_{n+1}) = \varphi(x, y_1) \wedge \cdots \wedge \varphi(x, y_{n+1})$, the new tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ and ψ witness the weak k-TP_1. This trick will be used in our proof of Proposition 33.

The equivalence of k-TP_1 and 2-TP_1 was proved in [6]. The following proposition in essence shows that the weak $(k+1)$-TP_1 implies the weak k-TP_1 unless there are many (independent) weak $(k+1)$-TP_1 trees.

Proposition 33. *Suppose that T has the weak $(k+1)$-TP_1, witnessed by the formula $\varphi(x, y)$. Then one of the following holds*:

1. *T has the weak k-TP_1, or*
2. *There are sets $I_i = (b_{i,\eta})_{\eta\in\omega^{<\omega}}$ $(i \in \omega)$ and a formula $\psi = \varphi(x, y_1) \wedge \cdots \wedge \varphi(x, y_m)$ with the following properties:*
   (a) *for each $i \in \omega$, $\{\psi(x, b_{i,\eta})\colon \eta \in \omega^{<\omega}\}$ witnesses the weak $(k+1)$-TP_1;*
   (b) *for each $i \in \omega$, let paths $\eta_{i,0}, \ldots, \eta_{i,k-1} \in \omega^\omega$ be given. Then $\bigcup_{i\in\omega}\{\psi(x, b_{i,\eta_{ij}|n})\colon j < k, n \in \omega\}$ is consistent.*

Proof. Let $\Gamma((y_\eta)_{\eta\in\omega^{<\omega}})$ be the set expressing that $\varphi(x, y)$ witnesses the weak $(k+1)$-TP_1. By Theorem 16, Γ is realized by a strongly indiscernible tree. Moreover, by compactness, there is a strongly indiscernible tree $A = (a_\eta)_{\eta\in\omega^{<\mathbb{Z}}}$ such that $(a_\eta)_{\eta\in\omega^{<\omega}}$ realizes Γ. For $\eta \in \omega^{<\mathbb{Z}}$, let η^* be the sequence defined by

$$\eta^*(i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ \eta(j) & \text{if } i = 2j + 1. \end{cases}$$

Then the mapping $\tau^* : \eta \mapsto \eta^*$ clearly preserves $\{<_{\mathrm{ini}}, <_{\mathrm{lex}}\}$-structure. Although τ^* does not preserve $\cap$, it has the following property

$$X \sim_0 Y \quad \Rightarrow \quad \tau^*(X) \sim_0 \tau^*(Y).$$

Let

$$B = (b_\eta)_{\eta\in\omega^{<\mathbb{Z}}} = (a_{\eta^*})_{\eta\in\omega^{<\mathbb{Z}}}.$$

Then, by the property of τ^* mentioned above, B is a strongly indiscernible tree. Since τ^* preserves the relation of being distant siblings, the L-formula $\varphi(x,y)$ and parameters $(b_\eta)_{\eta\in\omega^{<\omega}}$ also witness the weak $(k+1)$-TP_1. Then $(b_\eta)_{\eta\in\omega^{<\omega}}$ realizes Γ. Moreover, if $X=\langle 0\rangle, \langle 0,0\rangle, \langle 0,1\rangle$ and $Y=\langle 0\rangle, \langle 0,0,0\rangle, \langle 0,0,1\rangle$ then, although $X \not\sim_0 Y$, we have $\tau^*(X)\sim_0 \tau^*(Y)$. From this observation, we see that B has an additional property:

(*) Let $\nu\in\omega^{<\mathbb{Z}}$. For each $i=0,1$, let X_i be a family of distant siblings such that $\nu <_{\mathrm{ini}} X_i$. Then $\mathrm{tp}(b_\nu b_{X_0})=\mathrm{tp}(b_\nu b_{X_1})$.

(In the above, if Y is a set consisting of elements not bigger than ν (in the $<_{\mathrm{ini}}$-sense) then we also have $\mathrm{tp}(b_Y b_{X_0})=\mathrm{tp}(b_Y b_{X_1})$.) For each $i\in\omega$, let $\nu_i:\{k\in\mathbb{Z}:\ k\leqslant -i\}\to\omega$ be the sequence defined by

$$\nu_i(j)=\begin{cases}0 & \text{if } j<-i,\\ 1 & \text{if } j=-i.\end{cases}$$

Using ν_i, for each $\eta\in\omega^{<\omega}$, let $b_{i,\eta}=b_{\nu_i{}^\frown\eta}$. Now, for each $i\in\omega$, H_i will denote a k-element subset of ω^ω. Then there are two cases:

- for any such $(H_i)_{i\in\omega}$, $\bigcup_{i\in\omega}\{\varphi(x,b_{i,\eta|n}):\ \eta\in H_i\ n\in\omega\}$ is consistent, and
- there are $(H_i)_{i\in\omega}$ such that $\bigcup_{i\in\omega}\{\varphi(x,b_{i,\eta|n}):\ \eta\in H_i\ n\in\omega\}$ is inconsistent.

First assume the first case holds. Then, by the tree indiscernibility of B, each tree $(b_{i,\eta})_{\eta\in\omega^{<\omega}}$ realizes Γ. So, by the case assumption, we see that the conditions 2(a) and 2(b) are both satisfied.

We assume the second case. By compactness, there is a minimal finite set $F\subset\omega$ witnessing the second case. Then, by compactness again, choose minimal finite subsets $H_i'\subset H_i$ $(i\in F)$ such that $\bigcup_{i\in F}\{\varphi(x,b_{i,\eta|n}):\ \eta\in H_i',\ n\in\omega\}$ is inconsistent. Without loss of generality, because of strong indiscernibility, assume that there is $i\in F$ such that $|H_i'|\geqslant 2$. (If every H_i' is a singleton, then we replace ν_0 by $\nu_0\cap\nu_1$, and H_0' by $H_0'\cup H_1'$, and the new H_0' (i.e. $H_0'\cup H_1'$) has two elements.) Since other cases can be treated similarly (by the tree indiscernibility of B), we assume $F=\{0,1,\dots,l\}$ and $|H_0'|\geqslant 2$. By the minimality of H_i''s, for each path $\eta_0\in H_0$, the set

$$\{\varphi(x,b_{0,\eta_0|n}):\ n\in\omega\}\cup\bigcup_{i\in\{1,\dots,l\}}\{\varphi(x,b_{i,\eta|n}):\ \eta\in H_i',\ n\in\omega\}$$

is consistent. Let $X_0\subset\{\eta|n:\ \eta\in H_0',\ n\in\omega\}$ and $X_1\subset\{\eta|n:\ \eta\in H_1'\cup\cdots\cup H_l',\ n\in\omega\}$ be minimal finite sets such that $\bigcup_{i=0,1}\{\varphi(x,b_{i,\chi}):\ \chi\in X_i\}$ is inconsistent. Let $\gamma(x)=\bigwedge_{\chi\in X_1}\varphi(x,b_{1,\chi})$. We can always find $X_0'\subset X_0$ and $\nu\in\omega^{<\omega}$ with the following properties:

1. X_0' has at least two incomparable elements;
2. If $\chi,\chi'\in X_0'$ are incomparable, $\nu=\chi\cap\chi'$;
3. $\chi\cap\chi' <_{\mathrm{ini}} \nu$, for any $\chi\in X_0'$ and $\chi'\in Y$, where $Y=X_0\smallsetminus X_0'$.

Let $\delta(x)$ be the formula $\bigwedge_{\chi\in Y}\varphi(x,b_{0,\chi})$. Now we work inside the set defined by $\delta(x)\wedge\gamma(x)$, and regard the parameters in $\delta\wedge\gamma$ as constants. Then $\{\varphi(x,b_{0,\chi}):\ \chi\in X_0'\}$ is inconsistent. Applying a trick described in Remark 32 to the tree above ν, we may assume that X_0' is a set of distant siblings, by taking a new tree. Then X_0' has at most k elements, since H_0 has at most k paths. From this and the condition (*), we see that any k-element set $K\subset\omega^{<\omega}$ consisting of distant siblings, $\{\varphi(x,b_{0,\eta}):\ \eta\in K\}$ is inconsistent. Moreover, by the minimality of H_i''s and by the tree indiscernibility, $\{\varphi(x,b_{0,\eta|n}):\ n\in\omega\}$ is consistent for each path η. This shows that T has the weak k-TP_1, witnessed by $\varphi(x,y)\wedge\delta(x)\wedge\gamma(x)$ and the tree $(b_{0,\eta})_{\eta\in\omega^{<\omega}}$. □

5.3. Lowness

The notion of lowness was defined by Buechler in [2]. Let $\Sigma(x)$ be a set of formulas and $\varphi(x,y)$ a formula.

Definition 34. $D(\Sigma(x),\varphi(x,y))\geqslant 0$ if $\Sigma(x)$ is consistent. For a limit ordinal δ, $D(\Sigma(x),\varphi(x,y))\geqslant\delta$ if $D(\Sigma(x),\varphi(x,y))\geqslant\alpha$ for all $\alpha<\delta$. $D(\Sigma(x),\varphi(x,y))\geqslant\alpha+1$ if there is an indiscernible sequence $\{b_i:\ i\in\omega\}$ over $\mathrm{dom}(\Sigma)$ such that $D(\Sigma(x)\cup\{\varphi(x,b_i)\},\varphi(x,y))\geqslant\alpha$ $(i\in\omega)$, and $\{\varphi(x,b_i):\ i\in\omega\}$ is inconsistent. We say T is low if $D(x=x,\varphi(x,y))<\omega$ for any φ.

Definition 35. $D_{\mathrm{inp}}(\Sigma(x),\varphi(x,y))$ is the minimum cardinal κ for which there is no matrix $A=\{a_{ij}:\ (i,j)\in\kappa\times\omega\}$ such that (1) $\Sigma(x)\cup\{\varphi(x,a_{i\eta(i)}):\ i<\kappa\}$ is consistent $(\forall\eta\in\omega^\kappa)$, and (2) for all $i<\kappa$, $\{\varphi(x,a_{ij}):\ j\in\omega\}$ is k_i-inconsistent, for some $k_i\in\omega$.

Casanovas and Kim [3] showed the existence of a supersimple nonlow theory T. This T does not have infinitely many mutually independent partitions. However, there is a formula $\varphi(x,y)$ such that for each $k\in\omega$ we can find parameter sets

$A_i = \{a_{ij}\colon\ j \in \omega\}$ $(i < k)$ defining k independent partitions. More precisely, for this theory, we have $D_{\mathrm{inp}}(x = x, \varphi(x, y)) \leqslant \omega$ for any φ, and $D_{\mathrm{inp}}(x = x, \varphi(x, y)) = \omega$ for some φ. So it is natural to ask whether there is a simple nonlow theory T such that $D_{\mathrm{inp}}(x = x, \varphi(x, y)) < \omega$ for any φ. We prove that there is no such theory.

Proposition 36. *Suppose that T does not have TP_1. (Namely, T does not have k-TP_1 for any k. Simple theories satisfy this condition.) Then the following two conditions are equivalent*:

1. $D_{\mathrm{inp}}(x = x, \varphi(x, y)) < \omega$.
2. $D(x = x, \varphi(x, y)) < \omega$.

Proof. It is easy to check that $D_{\mathrm{inp}}(x = x, \varphi(x)) > k$ implies $D(x = x, \varphi(x)) > k$. So, it is sufficient to show the implication $(1 \to 2)$. Choose $k \in \omega$ with $D_{\mathrm{inp}}(x = x, \varphi(x, y)) = k$. By way of contradiction, we assume that $D(x = x, \varphi(x, y)) \geqslant \omega$. Fix $m \in \omega$. By $D(x = x, \varphi(x, y)) \geqslant \omega$, there is a set $A = \{a_\nu\colon\ \nu \in \omega^{2m}\}$ witnessing $D(x = x, \varphi(x, y)) \geqslant 2m$. We can assume that A is a weakly indiscernible tree. Then, A satisfies the following:

(a) $\{\varphi(x, a_{\eta|i})\colon\ i \leqslant 2m\}$ is consistent for any $\eta \in \omega^{2m}$;
(b) $\{\varphi(x, a_{\nu^\frown i})\colon\ i \in \omega\}$ is $k_{\mathrm{lh}(\nu)}$-inconsistent for any ν;
(c) For any $X \sim_s Y$ and $\psi(z)$, $\models \psi(a_X)$ if and only if $\models \psi(a_Y)$.

For $l < m$ and $\nu \in \omega^l$, we define

$$\nu_* = \big\langle \nu(0), 0, \nu(1), 0, \ldots, \nu(l-1), 0 \big\rangle \in \omega^{<2m}.$$

Let $X = \{\nu_0, \ldots, \nu_{k-1}\} \subset \omega^{<m}$ be a 2-$<_{\mathrm{ini}}$-incomparable set with $|X| = k$ and let $X_* = \{(\nu_0)_*, \ldots, (\nu_{k-1})_*\}$.

Claim A. $\{\varphi(x, a_{\nu_*})\colon\ \nu \in X\}$ *is inconsistent.*

Suppose this is not the case. Let $(\nu_i)_*^-$ be the immediate predecessor of $(\nu_i)_*$. For $\eta \in \omega^k$, let

$$Y_\eta = \big\{(\nu_i)_*^{-\frown}\langle \eta(i)\rangle\colon\ i < k\big\}.$$

By 2-incomparability of X, no distinct elements in X_* have the same parent. Therefore, $X_* \sim_s Y_\eta$ for all $\eta \in \omega^k$. Then, by the weak indiscernibility (the condition (c) above), the following Γ_η is also consistent, for each sequence $\eta = \langle m_0, \ldots, m_{k-1}\rangle$ of length k.

$$\Gamma_\eta = \big\{\varphi(x, a_{(\nu_0)_*^{-\frown}\langle m_0\rangle}), \ldots, \varphi(x, a_{(\nu_{k-1})_*^{-\frown}\langle m_{k-1}\rangle})\big\}.$$

On the other hand, by the condition (b), for each $l = 0, 1, \ldots, k-1$, the set

$$\big\{\varphi(x, a_{(\nu_i)_*^{-\frown}\langle n\rangle})\colon\ n \in \omega\big\}$$

is inconsistent ($k_{len((\nu_i)_*)}$-inconsistent). This yields $D_{\mathrm{inp}}(x = x, \varphi(x, z)) \geqslant k + 1$, a contradiction. (End of Proof of Claim.)

By Claim A, the set $\{\varphi(x, a_{\nu_*})\colon\ \nu \in \omega^m\}$ witnesses the k-TP_1 of height m. Since m was chosen arbitrarily, by compactness, we have a tree witnessing the k-TP_1, contradicting the assumption on T. □

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References

[1] J. Baldwin, S. Shelah, The stability spectrum for classes of atomic models, 2010, preprint.
[2] S. Buechler, Lascar strong types in some simple theories, J. Symbolic Logic 64 (2) (1999) 817–824.
[3] E. Casanovas, B. Kim, A supersimple nonlow theory, Notre Dame J. Form. Log. 39 (4) (1998) 507–518.
[4] M. Džamonja, S. Shelah, On $\lhd^*$-maximality (v.2), http://arxiv.org/abs/math/0009087, 2011, preprint.
[5] R. Grossberg, S. Shelah, A nonstructure theorem for an infinitary theory which has the unsuperstability property, Illinois J. Math. 30 (2) (1986) 364–390.
[6] B. Kim, H. Kim, Notions around the tree property 1, Ann. Pure Appl. Logic 162 (9) (2011) 698–709.
[7] B. Kim, H. Kim, L. Scow, Tree indiscernibilities, revisited, 2011, preprint.
[8] S. Shelah, Classification Theory, second edition, North-Holland, Amsterdam, 1990.
[9] F.O. Wagner, Simple Theories, Kluwer Academic Publishers, Dordrecht, 2000.